

A Path Integral Approach to Option Pricing with Stochastic Volatility: Some Exact Results

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Abstract

The Black-Scholes formula for pricing options on stocks and other securities has been generalized by Merton and Garman to the case when stock volatility is stochastic. The derivation of the price of a security derivative with stochastic volatility is reviewed starting from the first principles of finance. The equation of Merton and Garman is then recast using the path integration technique of theoretical physics. The price of the stock option is shown to be the analogue of the Schrodinger wavefunction of quantum mechanics and the exact Hamiltonian and Lagrangian of the system is obtained. The results of Hull and White are generalized to the case when stock price and volatility have non-zero correlation. Some exact results for pricing stock options for the general correlated case are derived.

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I Introduction

The problem of pricing the European call option has been well-studied starting from the pioneering work of Black and Scholes [1], [2]. The results of Black and Scholes were generalized by Merton, Garman [3] and others for the case of stochastic volatility and they derived a partial differential equation that the option price must satisfy.

The methods of theoretical physics have been applied with some success to the problem of option pricing by Bouchaud et al [4]. Analyzing the problem of option pricing from the point of view of physics brings a whole collection of new concepts to the field of mathematical finance as well as adds to it a set of powerful computational techniques. This paper is a continuation of applying the methodology of physics, in particular, that of path integral quantum mechanics, to the study of derivatives.

It should be noted that, unlike the paper by Bouchaud and Sornette [4] in which the pricing of derivatives is obtained by techniques which go beyond the conventional approach in finance, the present paper is based on the usual principles of finance. In particular, a continuous time random walk is assumed for the security and a risk-free portfolio is used to derive the derivative pricing equation; these are reviewed in Sec II. The main focus of this paper is to apply the computational tools of physics to the field of finance; the more challenging task of radically changing the conceptual framework of finance using concepts from physics is not attempted.

An explicit analytical solution for the pricing of the European call (or put) option for the case of stochastic volatility has so far remained an unsolved problem. Hull and White [5] studied this problem and obtained a series solution for the case when the correlation of stock price and volatility, namely ρ , is equal to zero, i.e. $\rho = 0$ (uncorrelated). Hull and White [5] also obtained an algorithm to numerically evaluate the option price using Monte Carlo methods for the case of $\rho \neq 0$.

Extensive numerical studies of option pricing for stochastic volatility based on the algorithm of Finucane [6] have been carried out by Mills et al [7].

The main focus and content of this paper is to study the problem of stochastic volatility from the point of view of the Feynman path integral [9]. The Feynman-Kac formula for the European call option is well known [2, 8]. There have also been some applications of path integrals in the study of option pricing [10, 11]. In this paper the path integral approach to option pricing is first discussed in its generality and then applied to the problem of stochastic

volatility.

The advantage of recasting the option pricing problem as a Feynman path integral is that this allows for a new point of view and which leads to new ways of obtaining solutions which are exact, approximate as well as numerical, to the pricing of options.

In Section II the differential equation of Merton and Garman [3] is derived from the principles of finance. In Section III this equation is recast in the formalism of quantum mechanics. In Section IV a discrete time path integral expression is derived for the option price with stochastic volatility which generalizes the Feynman-Kac formula. In Section V the path integration over the stochastic stock price is performed explicitly and a continuous time path integral is then obtained. And lastly in Section VI some conclusions are drawn.

II Security Derivative with Stochastic Volatility

We review the principles of finance which underpin the theory of security derivatives, and in particular that of the pricing of options. We will derive the results not using the usual method used in theoretical finance based on Ito-calculus (which we will review for completeness), but instead from the Langevin stochastic differential equation. Hence we start from first principles.

A security is any financial instrument which is traded in the capitals market; this could be the stock of a company, the index of a stock market, government bonds etc. A security derivative is a financial instrument which is derived from an underlying security and which is also traded in the capitals market. The three most widely used derivatives are options, futures and forwards; more complicated derivatives can be constructed out of these more basic derivatives.

In this paper, we analyze the option of an underlying security S . A European call option on S is a financial instrument which gives the owner of the option to buy or not to buy the security at some future time $T \geq t$ for the strike price of K .

At time $t = T$, when the option matures the value of the call option $f(T)$ is clearly given by

$$\begin{aligned} f(T, S(T)) &= \begin{cases} S(T) - K, & S(T) > K \\ 0, & S(T) < K \end{cases} & (1a) \\ &= g(S) & (1b) \end{aligned}$$

The problem of option pricing is the following: given the price of the security $S(t)$ at time t , what should be the price of the option f at time $t < T$? Clearly $f = f(t, S(t), K, T)$. This is a final value problem since the final value of f at $t = T$ has been specified and its initial value at time t needs to be evaluated.

The price of f will be determined by how the security $S(t)$ evolves in time. In theoretical finance, it is common to model $S(t)$ as a random (stochastic) variable with its evolution given by the stochastic Langevin equation (also called an Ito- Wiener process) as

$$\frac{dS(t)}{dt} = \phi S(t) + \sigma R(t) \quad (2a)$$

where R is the usual Gaussian white noise with zero mean; since white noise is assumed to be independent for each time t , we have the Dirac delta-function correlator given by

$$\langle R_t R_{t'} \rangle = \delta(t - t'). \quad (2b)$$

Note ϕ is the expected return on the security S and σ is its volatility. White noise $R(t)$ has the following important property . If we discretize time $t = n\epsilon$, then the probability distribution function of white noise is given by

$$P(R_t) = \sqrt{\frac{\epsilon}{2\pi}} e^{-\frac{\epsilon}{2} R_t^2} \quad (3a)$$

For random variable R it can be shown using eqn. (3a) that

$$R_t^2 = \frac{1}{\epsilon} + \text{random terms of } O(1) \quad (3b)$$

In other words, to leading order in ϵ , the random variable R_t^2 becomes deterministic. This property of white noise leads to a number of important results, and goes under the name of Ito calculus in probability theory.

As a warm-up, consider the case when $\sigma = \text{constant}$; this is the famous case considered by Black and Scholes. We have

$$\frac{df}{dt} = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon, S(t + \epsilon)) - f(t, S(t))}{\epsilon} \quad (4a)$$

or, using Taylors expansion

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \frac{dS}{dt} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \left\{ \frac{dS}{dt} \right\}^2 \epsilon + O(\epsilon^{1/2}) \quad (4b)$$

But

$$\begin{aligned} \left[\frac{dS}{dt}\right]^2 &= \sigma^2 S^2 R^2 + 0(1) \\ &= \frac{1}{\epsilon} \sigma^2 S^2 + 0(1) \end{aligned} \quad (4c)$$

Hence we have, for $\epsilon \rightarrow 0$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \phi S \frac{\partial f}{\partial S} + \sigma S \frac{\partial f}{\partial S} R \quad (4d)$$

Since eqn. (4d) is of central importance for the theory of security derivatives we also give a derivation of it based on Ito-calculus. Rewrite eqn. (2a) in terms differentials as

$$dS = \phi S dt + \sigma S dz$$

where the Wiener process $dz = R dt$, with $R(t)$ being the Gaussian white noise. Hence from eqn. (3b)

$$(dz)^2 = R_t^2 (dt)^2 = dt + 0(dt^{3/2})$$

and hence

$$(dS)^2 = \sigma^2 S^2 dt + 0(dt^{3/2}).$$

From the equations for dS and $(dS)^2$ given above we have

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2 + 0(dt^{3/2}) \\ &= \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dz \end{aligned}$$

and using $\frac{dz}{dt} = R$ we obtain eqn. (4d).

The fundamental idea of Black and Scholes is to form a portfolio such that, instantaneously, the change of the portfolio is independent of the white noise R . Consider the portfolio

$$\pi = f - \frac{\partial f}{\partial S} S \quad (5a)$$

i.e. π is a portfolio in which an investor sells an option f and buys $\frac{\partial f}{\partial S}$ amount of security S . We then have from eqns. (4b) and (5a)

$$\begin{aligned}
\frac{d\pi}{dt} &= \frac{df}{dt} - \frac{\partial f}{\partial S} \frac{dS}{dt} \\
&= \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}
\end{aligned}
\tag{5b}$$

The portfolio π has been adjusted so that its change is deterministic and hence is free from risk which comes from the stochastic nature of a security. This technique of cancelling the random fluctuations of one security (in this case of f) by another security (in this case S) is called hedging. The rate of return on π hence must equal the risk-free return given by the short-term risk-free interest rate r since otherwise one could arbitrage. Hence

$$\frac{d\pi}{dt} = r\pi \tag{5c}$$

which yields from eqn.(5b) the famous Black-Scholes [1] equation

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{5d}$$

Note the parameter ϕ of eqn. (2a) has dropped out of eqn. (5d) showing that a risk-neutral portfolio π is independent of the investors expectation as reflected in ϕ ; or equivalently, the pricing of the security derivative is based on a risk-neutral process which is independent of the investors opinion.

We now address the more complex case where the volatility $\sigma(t)$ is considered to be a stochastic (random) variable. We then have the following coupled Langevin equations (following Hull and White [5]), for $\sigma^2 = V$,

$$\frac{dS}{dt} = \phi S + \sigma S R \tag{6a}$$

$$\frac{dV}{dt} = \mu V + \xi V Q \tag{6b}$$

where μ is the expected rate of increase of the variance V , ξ its volatility and R & Q are correlated Gaussian white noise with zero means and with correlators

$$\langle R_t R_{t'} \rangle = \langle Q_t Q_{t'} \rangle = \delta(t - t') \tag{6c}$$

and

$$\langle R_t Q_{t'} \rangle = \rho \delta(t - t') \tag{6d}$$

with $\rho^2 < 1$ being the reflection of the correlation between S and V .

Note that one can choose a different process from eqn. (6) for stochastic volatility as has been done by various authors [5] but the main result of the derivation is not affected.

The procedure for deriving the equation for the price of an option is more involved when volatility is stochastic. Since volatility V is *not* a traded security this means that a portfolio can only have two instruments, namely f and S . To hedge away the all fluctuations due to both S and V , we need at least *three* financial instruments to make a portfolio free from the investors subjective preferences. Since we have only two instruments from which to form our portfolio we will not be able to completely avoid the subjective expectations of the investors.

Consider the change in the pricing of the option in the presence of stochastic volatility; using eqns (6a) and (6b) we have from Taylors expansion

$$\begin{aligned}\frac{df}{dt} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{f(t + \epsilon, V(t + \epsilon), S(t + \epsilon)) - f(t, V(t), S(t))\} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(\phi S + \sigma S R) + \frac{\partial f}{\partial V}(\mu V + \xi V Q) \\ &\quad + \frac{\epsilon}{2} \left\{ \frac{\partial^2 f}{\partial S^2} \left[\frac{dS}{dt} \right]^2 + \frac{\partial^2 f}{\partial V^2} \left[\frac{dV}{dt} \right]^2 + 2 \frac{\partial^2 f}{\partial S \partial V} \frac{dV}{dt} \frac{dS}{dt} \right\} + o(\epsilon^{1/2})\end{aligned}\tag{7a}$$

Using

$$\begin{aligned}R_t^2 &= Q_t^2 = +o(1), \\ R_t Q_t &= \frac{\rho}{\epsilon} + o(1)\end{aligned}\tag{7b}$$

we have, similar to eqn. (4a), for $\epsilon \rightarrow 0$

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + \mu V \frac{\partial f}{\partial V} + \frac{1}{2} \left[\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \zeta^2 V^2 \frac{\partial^2 f}{\partial V^2} + \xi \sigma^3 S \frac{\partial^2 f}{\partial V \partial S} \right] \\ &\quad + \sigma S \frac{\partial f}{\partial S} R + \epsilon V \frac{\partial f}{\partial V} Q\end{aligned}\tag{7c}$$

Define the deterministic and stochastic terms by

$$\frac{df}{dt} \equiv f\omega + f\alpha_1 R + f\alpha_2 Q\tag{7d}$$

Consider a portfolio which consists of $\theta_1(t)$ amount of option f and $\theta_2(t)$ amount of stock S , that is

$$\pi(t) = \theta_1(t)f(t) + \theta_2(t)S(t)\tag{8a}$$

Then

$$\frac{d\pi}{dt} = \theta_1 \frac{df}{dt} + \theta_2 \frac{dS}{dt} + \frac{d\theta_1}{dt} f + \frac{d\theta_2}{dt} S \quad (8b)$$

We consider a self-replicating portfolio in which no cash flows in or out and the quantities $\theta_1(t)$ and $\theta_2(t)$ are adjusted accordingly. It then follows that

$$\frac{d\pi}{dt} = \theta_1 \frac{df}{dt} + \theta_2 \frac{dS}{dt} \quad (8c)$$

Hence

$$\frac{d\pi}{dt} = \theta_1 f \omega + \phi \theta_2 S + (\alpha_1 R + \alpha_2 Q) \theta_1 f + \theta_2 S \sigma R \quad (8d)$$

We choose the portfolio such that, in matrix notation

$$\begin{bmatrix} \alpha_1 & \sigma \\ \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 f \\ \theta_2 S \end{bmatrix} = 0 \quad (8e)$$

which yields

$$\frac{d\pi}{dt} = \theta_1 f \omega + \phi \theta_2 S \quad (9a)$$

Note that all the random terms from the rhs of eqn. (8d) have been removed by our choice of portfolio which satisfies the constraint given in eqn. (8e). Since $\frac{d\pi}{dt}$ is now deterministic, the absence of arbitrage requires

$$\frac{d\pi}{dt} = r\pi \quad (9b)$$

or, from eqns. (8d) and (9a)

$$(\omega - r)\theta_1 f + (\phi - r)\theta_2 S = 0 \quad (9c)$$

We hence have from eqns. (8e) and (9c) that eqn. (9c) above can be satisfied in general by making the following choice ω and ϕ , namely

$$[\omega - r, \phi - r] = [\lambda_1(t), \lambda_2(t)] \begin{bmatrix} \alpha_1 & \sigma \\ \alpha_2 & 0 \end{bmatrix}$$

where λ_1 and λ_2 are arbitrary. In components, we have from above

$$\omega - r = \lambda_1 \alpha_1 + \lambda_2 \alpha_2 \quad (9d)$$

$$\phi - r = \lambda_1 \sigma \quad (9e)$$

Eliminating λ_1 yields using eqn. (9e) yields from eqn. (9d)

$$f\omega - rf = \left[\frac{\phi - r}{\sigma} \right] + \lambda_2 f \alpha_2$$

and hence from eqns. (7c) and (7d) we obtain the Merton and Garman [3] equation given by

$$\frac{\partial f}{\partial t} + \frac{1}{2} \left[\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\xi\sigma^3 S \frac{\partial^2 f}{\partial S \partial V} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} \right] - rf = -rS \frac{\partial f}{\partial S} + [\lambda_2 - \mu] V \frac{\partial f}{\partial V} \quad (10a)$$

Note the fact that both ϕ and λ_1 have been eliminated from (10a) since we could hedge for the fluctuations of S by including it in the portfolio π . In other words the subjective view of the investor vis-a-vis security S has been removed. However, the appearance of λ_2 in eqn. (10a) reflects the view of the investor of what he expects for the volatility of S . Since V is not traded, the investor can't execute a perfect hedge against it as was the case for constant volatility and hence the pricing of the option given in eqn. (10a) has a subjective factor in it.

It has been argued by Hull and White [5] that for a large class of problems, we can consider λ to be a constant which in effect simply redefines μ to be $\mu - \lambda_2$.

Hence we have

$$\frac{\partial f}{\partial t} + \frac{1}{2} \left[\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\xi\sigma^3 S \frac{\partial^2 f}{\partial S \partial V} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} \right] - rf = -rS \frac{\partial f}{\partial S} - \mu\sigma^2 \frac{\partial f}{\partial V} \quad (10b)$$

Eqn. (10) above is an equation for *any* security derivative with stochastic volatility. Whether it is an European option or an Asian option or an American option is determined by the *boundary conditions* that are imposed on f ; in particular the boundary condition given in eqn. (1) is that of an European call option.

III Black-Scholes-Schrodinger Equation for Option Price

We recast the Merton and Garman equation given in eqn. (10) in terms of the formalism of quantum mechanics; in particular we derive the Hamiltonian for eqn. (10) which is the generator of infinitesimal translations in time. Since both S and V are random variables which can never take negative values, we define new variables x and y

$$S = e^x, -\infty < x < \infty \quad (11a)$$

$$V = e^y, -\infty < y < \infty. \quad (11b)$$

From eqn. (10b), we then obtain the Black-Scholes-Schrodinger equation for the price of security derivative as

$$\frac{\partial f}{\partial t} = (H + r)f \quad (12a)$$

where f is the Schrodinger wave function. The ‘Hamiltonian’ H is a differential operator given by

$$H = -\frac{e^Y}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}e^Y - r\right) \frac{\partial}{\partial x} - \xi \rho e^{Y/2} \frac{\partial^2}{\partial x \partial y} - \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2} + \left(\frac{1}{2}\xi^2 - \mu\right) \frac{\partial}{\partial y} \quad (12b)$$

From eq. (12a) we have the formal solution

$$f = e^{t(H+r)} \tilde{f} \quad (13)$$

where \tilde{f} is fixed by the boundary condition. Note from eqn. (13) that option price f seems to be unstable being represented by a growing exponential; however, as will be seen later since the boundary condition for f is given at final time T eqn. (13) will be converted to a decaying exponential in terms of remaining time $\tau = T - t$.

Let $p(x, y, \tau | x', y')$, be the conditional probability that, given security price x' and volatility y' at time $\tau = 0$, it will have a value of x and volatility y at time t . We have the boundary condition at $\tau = 0$ given by Dirac delta-functions, namely

$$p(x, y, 0 | x', y') = \delta(x' - x) \delta(y' - y) \quad (14)$$

The derivative price is then given, for $0 \leq t \leq T$, by the Feynman-Kac formula Refs [5, 8] as

$$f(t; x, y) = e^{-(T-t)r} \int_{-\infty}^{+\infty} dx' p(x, y, T - tx') g(x') \quad (15a)$$

where

$$p(x, y, T - tx') = \int_{-\infty}^{+\infty} dy' p(x, y, T - tx', y') \quad (15b)$$

Note that integration over y' is decoupled from the function $g(x')$ due to eqn. (1). It follows from eqn. (14) that $f(t; x, y)$ given in (15a) satisfies the boundary condition given in eqn. (1), i.e.

$$f(T, x, y) = g(x) \quad (15b)$$

We rewrite eqns. (15a) and (15b) in the notation of quantum mechanics. The function $f(x)$ can be thought of as an infinite dimensional vector $|f\rangle$ of a function space with components $f(x) = \langle x|f\rangle$ and $f^*(x) = \langle f|x\rangle$ (where $*$ stands for complex conjugation). For $-\infty \leq x \leq \infty$ the bra vector $\langle x|$ is the basis of the function space and the ket vector $|x\rangle$ is its dual with normalization

$$\langle x|x'\rangle = \delta(x - x'). \quad (16)$$

The scalar product of two functions is given by

$$\begin{aligned} \langle f|g\rangle &\equiv \int dx f^*(x)g(x) \\ &= \langle f|\{\int_{-\infty}^{\infty} dx |x\rangle\langle x|\}|g\rangle \end{aligned}$$

and yields the completeness equation

$$I = \int_{-\infty}^{\infty} dx |x\rangle\langle x| \quad (17a)$$

where I is the identity operator on function space. For the case of stochastic stock price and stochastic volatility, we have

$$I = \int_{-\infty}^{\infty} dx dy |x, y\rangle\langle x, y| \quad (17b)$$

where $|x, y\rangle \equiv |x\rangle \otimes |y\rangle$.

From eqn (13) we have, in Dirac's notation

$$|f, t\rangle = e^{t(H+r)}|f, 0\rangle$$

and boundary condition given in eqn. (1b) yields

$$|f, T\rangle = e^{t(H+r)}|f, 0\rangle = |g\rangle.$$

Hence

$$|f, t\rangle = e^{(T-t)(H+r)}|g\rangle \quad (18a)$$

or, more explicitly

$$\begin{aligned} f(t, x, y) &= \langle x, y|f, t\rangle \\ &= e^{-r(T-t)} \langle x, y|e^{-(T-t)H}|g\rangle \end{aligned} \quad (18b)$$

or, using completeness equation (17b)

$$f(t, x, y) = e^{-r(T-t)} \int_{-\infty}^{\infty} dx dy' \langle x, y | e^{-(T-t)H} | x', y' \rangle g(x') \quad (19a)$$

which yields for remaining time $\tau = T - t$, from eqn. (15b)

$$p(x, y, \tau | x', y') = \langle x, y | e^{-\tau H} | x', y' \rangle. \quad (19b)$$

Note remaining time runs backwards, i.e. when $\tau = 0$, we have $\tau = T$ and when $\tau = T$, real time $t = 0$; as mentioned earlier, expressed in terms of remaining time τ the derivative price is given by a decaying exponential as given in eqn. (19b).

Before tackling the more complicated case of option pricing with stochastic volatility I apply the formalism developed so far to the simpler case of constant volatility; the detailed derivation is given in Appendix 1. The well known results of Black and Scholes are seen to emerge quite naturally in this formalism.

IV. The Discrete-Time Feynman Path Integral

The Feynman path integral [9] is a formulation of quantum mechanics which is based on functional integration. In particular, the Feynman path integral provides a functional integral realization of the conditional probability $p(x, y, \tau | x', y')$. To obtain the path integral we discretize time τ into N points with spacing $\epsilon = \tau/N$. Then, from (19b),

$$p(x, y, \tau | x', y') = \lim_{N \rightarrow \infty} \langle x, y | e^{-\epsilon H} \cdots e^{-\epsilon H} | x', y' \rangle. \quad (20)$$

Hence forth, we will always assume the $N \rightarrow \infty$ limit.

Inserting the completeness equation for x and y , namely eqn (17b), $(N - 1)$ times in eqn.(20) yields

$$p(x, y, \tau | x', y') = \left(\prod_{i=1}^{N-1} \int dx_i dy_i \right) \prod_{i=1}^N \langle x_i, y_i | e^{-\epsilon H} | x_{i-1}, y_{i-1} \rangle \quad (21a)$$

with boundary conditions

$$x_0 = x', \quad y_0 = y' \quad (21b)$$

$$x_N = x, \quad y_N = y. \quad (21c)$$

We show in the Appendix 2 that for the H given by eqn (12b) we have the Feynman relation given in (A9) and (A13)

$$\langle x_i, y_i | e^{-\epsilon H} | x_{i-1}, y_{i-1} \rangle = \frac{1}{2\pi\epsilon} \frac{e^{-y_i/2}}{\xi \sqrt{1-\rho^2}} e^{\epsilon L(i)} \quad (22a)$$

where the ‘lagrangian’ L is given in eqn (A15), for $\delta x \equiv x_i - x_{i-1}$ and $\delta y \equiv y_i - y_{i-1}$ as

$$L(i) = -\frac{1}{2\xi^2} \left(\frac{\delta y_i}{\epsilon} + \mu - \frac{1}{2}\xi^2 \right)^2 - \frac{e^{-y_i}}{2(1-\rho^2)} \left\{ \frac{\delta x_i}{\epsilon} + r - \frac{1}{2}e^{y_i} - \frac{\rho}{\xi} e^{y_i/2} \left(\frac{\delta y_i}{\epsilon} + \mu - \frac{1}{2}\xi^2 \right) \right\}^2 + O(\epsilon) \quad (22b)$$

For $\epsilon \rightarrow 0$, we have, as expected

$$\langle x_i, y_i | e^{-\epsilon H} | x_{i-1}, y_{i-1} \rangle = \delta(x_i - x_{i-1}) \delta(y_i - y_{i-1}) + O(\epsilon) \quad (23)$$

and the prefactors to the exponential on the rhs of (22a) ensure the correct limit.

We form the ‘action’ S by

$$S = \epsilon \sum_{i=1}^N L(i) + O(\epsilon). \quad (24)$$

Note S is quadratic in x_i and non-linear in the y_i variables.

Hence, from eqns. (20), (21), (22) and (24) we have the path integral

$$p(x, y, \tau | x') = \int_{-\infty}^{\infty} dy' p(x, y, \tau | x', y') \quad (25a)$$

$$= \lim_{N \rightarrow \infty} \int DX DY e^S \quad (25b)$$

where, for $\epsilon = \tau/N$

$$\int DX = \frac{e^{-Y_N/2}}{\sqrt{2\pi\epsilon(1-\rho^2)}} \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} \frac{dx_i e^{-y_i/2}}{\sqrt{2\pi\epsilon(1-\rho^2)}} \quad (25c)$$

and

$$\int DY = \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} \frac{dy_i}{\sqrt{2\pi\epsilon\xi^2}}. \quad (25d)$$

Note we have included an extra $dy_0 = dy'$ integration in DY given in (25d) due to the integration over y' in eqn (25a).

Eqn (25) is the path integral for stochastic stock price with stochastic volatility in its full generality. To evaluate expressions such as the correlation of $S(t)$ with $V(t')$, the path integral in eqn (25) has to be used.

Since the action S is quadratic in the x_i , the path integral over the x_i variables in eqn. (25) can be done exactly. The details are given in Appendix 3.

From eqn (A21) we obtain

$$p(x, y, \tau | x') = \int DY \frac{e^{S_0 + S_1}}{\sqrt{2\pi\epsilon(1 - \rho^2) \sum_{i=1}^N e^{y_i}}} \quad (26a)$$

where the first term in eqn. (22b) gives

$$S_0 = -\frac{\epsilon}{2\xi^2} \sum_{i=1}^N \left(\frac{\delta y_i}{\epsilon} + \mu - \frac{\xi^2}{2} \right)^2 \quad (26b)$$

and S_1 is the result of the DX path integration given by eqn (A21b) as

$$S_1 = -\frac{1}{2(1 - \rho^2)\epsilon \sum_{i=1}^N e^{y_i}} \left\{ x - x' + \epsilon \sum_{i=1}^N \left(r - \frac{1}{2} e^{y_i} \right) - \frac{\rho}{\xi} \sum_{i=1}^N e^{y_i/2} [\delta y_i + \epsilon(\mu - \frac{\xi^2}{2})] \right\}^2. \quad (26c)$$

Extensive numerical studies of the pricing of European call option has been carried out in [13] based on eqns. (26a-c).

V. Continuous Time Feynman Path Integral

We first derive the continuum limit of the Black-Scholes constant volatility case before analyzing the more complex case of stochastic volatility. The DX_{BS} - path integral for the Black-Scholes case, from eqn (A5b), has a measure $DX_{BS} = [\frac{1}{2\pi\epsilon\sigma^2}]^{N/2} \prod_{i=1}^N dx_i$ which is essentially the measure for the flat space \mathbf{R}^N ; we can hence take the $N \rightarrow \infty$ for DX and obtain a well defined continuous-time path integral [9]. From eqn (A5a) we obtain taking the continuum limit of $\epsilon \rightarrow 0$

$$S_{BS} = \int_0^\tau dt L_{BS} = -\frac{1}{2\sigma^2} \int_0^\tau dt \left(\frac{dx}{dt} + r - \frac{1}{2}\sigma^2 \right)^2$$

with boundary conditions $x(0) = x'$ and $x(\tau) = x$. We see from above that the Black-Scholes case of constant volatility corresponds to the evolution of a free quantum mechanical particle with mass given by $1/\sigma^2$. This yields the Black-Scholes result

$$p_{BS}(x, \tau | x') = \langle x | e^{-\tau H_{BS}} | x' \rangle = \int DX_{BS} e^{S_{BS}}$$

The exact result for $p_{BS}(x, \tau|x')$ is given in eqn (A4).

Eqn. (26) provides a mathematically rigorous basis for taking the $N \rightarrow \infty$ limit for the case of stochastic volatility. On exactly performing the $\int DX$ -path integral, the remaining $\int DY$ - path integral has a measure $DY = [\frac{1}{2\pi\epsilon\xi^2}]^{N/2} \prod_{i=1}^N dy_i$ which, just as in the Black-Scholes case, is essentially the measure for the flat space \mathbb{R}^N (unlike the nonlinear expression in eqn. (25c)); we can hence take the $N \rightarrow \infty$ for DY and for $S_0 + S_1$ and obtain a well defined continuous-time path integral.

We take the limit of $\epsilon \rightarrow 0$; we have, $t = i\epsilon$, $\epsilon \sum_{i=1}^N e^{Y_i} \rightarrow \int_0^\tau dt e^{Y(t)} \equiv \tau w$, and $\frac{1}{\epsilon} \delta y_i \rightarrow \frac{dy}{dt}$. Hence, from (26) we have

$$S = S_0 + S_1 \quad (27a)$$

$$S_0 = -\frac{1}{2\xi^2} \int_0^\tau dt \left(\frac{dy}{dt} + \mu - \frac{1}{2}\xi^2 \right)^2 \quad (27b)$$

$$S_1 = -\frac{1}{2(1-\rho^2)w} \left\{ x - x' + r\tau - \frac{1}{2} \int_0^\tau dt e^{Y(t)} \right. \\ \left. + \frac{2\rho}{\xi} (e^{Y(0)/2} - e^{Y/2}) - \frac{\rho}{\xi} \left(\mu - \frac{\xi^2}{2} \right) \int_0^\tau dt e^{Y(t)/2} \right\}^2 \quad (27c)$$

with boundary value

$$y(\tau) = y \quad (27d)$$

and

$$p(x, y, \tau|x') = \int DY \frac{e^S}{\sqrt{2\pi(1-\rho^2)\tau w}}. \quad (28)$$

Note taking the continuum limit is possible only after the discrete $\int DX$ path integration has been performed since the nonlinear measure given by (25c) does not have a finite continuum limit. All correlation functions $\langle S(t)V(t') \rangle$ can be obtained from the continuum limit of the discrete correlators $\langle e^{X_n} e^{Y_m} \rangle$.

For the case of $\rho = 0$, to recover the Black-Scholes formula, set $w = \sigma^2$, where $\sigma = e^{Y/2}$ is the volatility at time $t = 0$. As has been noted by Hull and White [5], for the case of $\rho = 0$ the conditional probability depends on stochastic volatility $y(t)$ only through the combination $w = \frac{1}{\tau} \int_0^\tau e^{Y(t)/2} dt$. For $\rho \neq 0$, we see from eqn. (27c) that $p(x, y, \tau|x')$ now depends on w as well as on $u = \frac{1}{\tau} \int_0^\tau e^{Y(t)/2} dt$ and $e^{Y(0)/2}$. We hence have from (27) and (28)

$$p(x, y, \tau|x') = \int_0^\infty dw \int du \int dv \frac{e^{S_1(u,v,w)}}{\sqrt{2\pi(1-\rho^2)\tau w}} g(u, v, w) \quad (29a)$$

where

$$S_1(u, v, w) = -\frac{1}{2(1-\rho^2)\tau w} \left\{ x - x' + r\tau - \frac{\tau}{2}w + \frac{2\rho}{\xi}(v - e^{Y/2}) - \frac{\rho}{\xi}(\mu - \frac{\xi^2}{2})u \right\}^2. \quad (29b)$$

and $g(u, v, w)$ is given by the path integral

$$g(u, v, w) = \int DY e^{S_0} \delta\{w - \frac{1}{\tau} \int_0^\tau e^{Y(t)} dt\} \delta\{v - e^{Y(\tau)/2}\} \delta\{u - \frac{1}{\tau} \int_0^\tau e^{Y(t)/2} dt\} \quad (29c)$$

From eqn (29c) we see that $g(u, v, w)$ is the probability density for u, v , and w . The path integral for $g(u, v, w)$ is nonlinear and cannot be performed exactly. We see that $p(x, y, \tau|x')$ is the weighted average of the integrand $e^{S_1(u, v, w)}/\sqrt{2\pi(1-\rho^2)\tau w}$ with respect to $g(u, v, w)$.

Note $g(u, v, w)$ has the remarkable property that it is independent of ρ . Following Hull and White [5] one can expand the integrand in (29a) in an infinite power series in u, v, w and reduce the evaluation of $p(x, y, \tau|x')$ to finding all the moments of u, v and w ; in other words we need to evaluate

$$\langle u^n w^m v^p \rangle \equiv \int_0^\infty du \, dw \, u^n w^m v^p g(u, v, w) \quad (30a)$$

$$= \int DY \left[\frac{1}{\tau} \int_0^\tau e^{Y(t)/2} dt \right]^n \left[\frac{1}{\tau} \int_0^\tau e^{Y(t)} dt \right]^m e^{pY(0)/2} e^{S_0} \quad (30b)$$

The path integral for $\langle u^n w^m v^p \rangle$ can be performed exactly. Rewrite (30b) as

$$\langle u^n w^m v^p \rangle = \frac{1}{\tau^{n+m}} \int_0^\tau dt_1 \cdots dt_n dt_{n+1} \cdots dt_{n+m} Z(j, y, p) \quad (30c)$$

where

$$Z(j, y, p) = \int DY \exp \left[\int_0^\tau dt j(t) y(t) \right] e^{pY(0)/2} e^{S_0} \quad (31a)$$

and from eqns. (30b) and (30c) we have

$$j(t) = \frac{1}{2} \sum_{i=1}^n \delta(t - t_i) + \sum_{i=n+1}^{n+m} \delta(t - t_i) + \frac{p}{2} \delta(t) \quad (31b)$$

$$\equiv \sum_{i=1}^{n+m} a_i \delta(t - t_i) + \frac{p}{2} \delta(t). \quad (31c)$$

The path integral for $Z(j, y, p)$ is evaluated exactly in Appendix 3 and yields

$$\langle u^n w^m v^p \rangle = \sigma^p \frac{e^{\sum_i a_i Y}}{\tau^{n+m}} \prod_{i=1}^{n+m} \int_0^\tau dt_i e^F \quad (32a)$$

where from eqns (A41a), (A41b) and (31c) we have, after some simplifications

$$F = (\mu - \frac{1}{2}\xi^2) \sum_{i=1}^{n+m} a_i t_i + \xi^2 \sum_{i,j=1}^{n+m} a_i a_j \theta(t_i - t_j) t_j + F' \quad (32b)$$

with

$$F' = \frac{1}{2}p\tau(\mu - \frac{1}{2}\xi^2) + \frac{1}{2}p\xi^2 \sum_{i=1}^{n+m} a_i t_i + \frac{1}{8}p^2\xi^2\tau. \quad (32c)$$

Note that in obtaining eqn (32) we have used the identity that $\theta(0) = \frac{1}{2}$, as given in eqn. (A37a) and reflects the identity $\int_0^\tau dt \delta(t - \tau) = \frac{1}{2}$. All the t_i integrations in (32) can be performed exactly since the exponent is linear in the t_i 's. The expression (32) for $\langle u^n w^m v^p \rangle$ generalizes the result of Hull and White [5] as we have an exact expression for all the moments of u, v and w as well as for their cross-correlators.

We explicitly evaluate the first few moments using eqn (32).

$$\begin{aligned} \langle w \rangle &= \frac{e^Y}{\tau} \int_0^\tau dt e^{\mu t} \\ &= \frac{V}{\mu\tau} (e^{\mu\tau} - 1) \end{aligned} \quad (33)$$

where $V = e^Y$;

$$\begin{aligned} \langle w^2 \rangle &= \frac{2e^{2Y}}{\tau^2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 e^{\mu(t_1+t_2)} e^{\xi^2 t_2} \\ &= \frac{2V^2}{\tau^2} \left[\frac{e^{(2\mu+\xi^2)\tau}}{(\xi^2 + \mu)(2\mu + \xi^2)} - \frac{e^{\mu\tau}}{\mu(\mu + \xi^2)} + \frac{1}{\mu(2\mu + \xi^2)} \right]. \end{aligned} \quad (34)$$

We evaluate $\langle w^3 \rangle$ for the case of $\mu = 0$; we have from eqn (32)

$$\begin{aligned} \langle w^3 \rangle &= \frac{6e^{3Y}}{\tau^3} \int_0^\tau dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 e^{\xi^2(t_2+2t_3)} \\ &= \frac{V^3}{3\xi^6\tau^2} (e^{3\xi^2\tau} - 9e^{\xi^2\tau} + 8 + 6\xi^2\tau) \end{aligned} \quad (35)$$

Eqns (33), (34) and (35) agree exactly with the results stated (without derivation) in Hull & White [5]. It is reassuring to see that two very different formalisms agree exactly and this increases ones confidence in the path integral approach.

We compute a few moments that have not been given in Hull and White [5]. We further have (recall $\sigma = e^{Y/2}$)

$$\begin{aligned}
\langle u \rangle &= \frac{e^{Y/2}}{\tau} \int_0^\tau dt e^{(\mu/2 - \xi^2/8)t} \\
&= \frac{2\sigma}{\tau(\mu - \xi^2/4)} \left[e^{(\mu/2 - \xi^2/8)\tau} - 1 \right]. \tag{36a}
\end{aligned}$$

$$\langle u^2 \rangle = \frac{2e^Y}{\tau^2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 e^{\mu(t_1+t_2)/2} e^{-\xi^2(t_1-t_2)/8} \tag{36b}$$

$$= \frac{4V}{\tau^2} \left[\frac{e^{(\mu/2 - \xi^2/8)\tau}}{(\mu^2 + \xi^2/4)(\mu - \xi^2/8)} - \frac{2e^{(\mu/2 - \xi^2/4)\tau}}{(\mu + \xi^2/4)(\mu - \xi^2/2)} - \frac{1}{(\mu - \xi^2/2)(\mu - \xi^2/8)} \right] \tag{37}$$

and lastly, for $a_1 = \frac{1}{2}$, $a_2 = 1$ in eqn (31c), we have

$$\begin{aligned}
\langle uw \rangle &= \frac{2e^{3Y/2}}{\tau^2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 e^{\mu(t_1/2+t_2)} e^{\xi^2(t_1-6t_2)/8} \\
&= \frac{4\sigma V}{\tau^2} \left[\frac{e^{(3\mu/2+3\xi^2/8)\tau}}{3(\mu + \xi^2/2)(\mu + \xi^2/4)} - \frac{e^{(\mu/2 - \xi^2/8)\tau}}{(\mu + \xi^2/2)(\mu^2 - \xi^2/4)} + \frac{1}{3(\mu - \xi^4/16)} \right]. \tag{38}
\end{aligned}$$

All the other moments $\langle u^n w^m v^p \rangle$ can similarly be evaluated, which in turn yields an infinite series solution for $p(x, y, \tau|x')$ in eqn (29a).

VI. Conclusion

The complete information regarding the dynamics of how stock price $S(t)$ and its volatility $V(t)$ evolve, their cross-correlators as well as their fluctuations is given by the discrete time path integral given in eqn. (25). In particular, to numerically study correlators such as $\langle e^{X_n} e^{Y_m} \rangle$ we need to start from the path integral given in eqn. (25).

However, if one is interested solely in the price of the option, one needs to only determine $p(x, y, \tau|x', y')$ and in this case the simplified discrete-time path integral obtained in eqn. (26) should be used as the starting point for numerical studies. A detailed numerical simulation based on the results of this paper has been carried out in ref [13]; in particular, it has been shown that implied volatility smile and frown can be obtained for different values of the correlation parameter ρ as well as for varying values of $S(t)$.

In summary, we reformulated the option pricing problem in the language of quantum mechanics. We then obtained a number of exact results for the option price of a security

derivative using the path integral method. Stochastic volatility introduces a high order of nonlinearity in the option pricing problem and needs to be studied using approximate and numerical techniques.

The continuous time path integral given in eqn. (29c) is a nonlinear path integral which has no known exact solution. Various techniques could be used to obtain new approximations for $g(u,v,w)$ relevant for special applications.

Numerical algorithms based on the path integral provide a wide range of new numerical algorithms which go beyond the usual binomial tree and its generalizations.

The comparative efficiency of the different algorithms is being studied. The Feynman path integral has been numerically studied using many algorithms and which could prove useful in addressing problems in finance.

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Appendix 1 Black-Scholes Formula for Constant Volatility

For the case of constant volatility the Black-Scholes equation is given by the following [2]

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rS \frac{\partial f}{\partial S} + rf \quad (A1)$$

Making the change of variable as in (2)

$$S = e^x, \quad -\infty \leq x \leq \infty$$

yields

$$\frac{\partial f}{\partial t} = (H + r)f$$

with the Black-Scholes Hamiltonian given by

$$H_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right) \frac{\partial}{\partial x} \quad (A2)$$

To evaluate the price of the European call option with constant volatility we have the Feynman-Kac formula

$$f(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} dx' \langle x | e^{-(T-t)H_{BS}} | x' \rangle g(x')$$

To compute $p_{BS}(x, \tau | x') = \langle x | e^{-\tau H_{BS}} | x' \rangle$, where remaining time is $\tau = T - t$, we use the Hamiltonian approach to illustrate another powerful technique of quantum mechanics. We go to the 'momentum' basis in which H_{BS} is diagonal. The Fourier transform of the $|x\rangle$ basis to 'momentum space' is given by

$$\begin{aligned} \langle x | x' \rangle &= \delta(x - x') = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')} \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x | p \rangle \langle p | x' \rangle \end{aligned}$$

which yields for momentum space basis $|p\rangle$ the completeness equation

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi} |p\rangle \langle p| \quad (A3)$$

and the scalar product

$$\langle x | p \rangle = e^{ipx}; \quad \langle p | x \rangle = e^{-ipx}.$$

From (A2) and the equation above we have the matrix elements of H given by

$$\langle x|H_{BS}|p\rangle = H_{BS}e^{ipx} = \left\{\frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p\right\}e^{ipx}$$

Using (A3) and the equation above yields

$$\begin{aligned}\langle x|e^{-\tau H_{BS}}|x'\rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|e^{-\tau H_{BS}}|p\rangle \langle p|x'\rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left(-\frac{1}{2}\tau\sigma^2 p^2\right) e^{ip(x-x'+\tau(r-\sigma^2/2))} \\ &= \frac{1}{\sqrt{2\pi\tau\sigma^2}} \exp\left[-\frac{1}{2\tau\sigma^2}\{x-x'+\tau(r-\sigma^2/2)\}^2\right]\end{aligned}\tag{A4}$$

The result above is the Black-Scholes distribution. Recall $x' = \log(S(T))$, $x = \log(S(t))$ and $\tau = T - t$; the stock price evolves randomly from its given value of $S(t)$ at time t to a whole range of values for $S(T)$ at time T . Equation (A4) above states that $\log(S(T))$ has a normal distribution with mean equal to $\log(S(t)) + (r - \sigma^2/2)(T - t)$ and variance of $\sigma^2(T - t)$ as is expected for the Black-Scholes case with constant volatility [2].

In general for a more complicated (nonlinear) Hamiltonian such as the one given in eqn. (12b) for stochastic volatility it is not possible to exactly diagonalize H and then be able to exactly evaluate the matrix elements of $e^{-\tau H}$. The Feynman path integral is an efficient theoretical tool for analysing such nonlinear Hamiltonians, and this is the reason for using the path integral formalism.

Appendix 2 The Lagrangian

The lagrangian L is central to the path integral formulation of quantum mechanics. We first find L_{BS} for the Black-Scholes case of constant volatility before tackling the more complicated case. For infinitesimal time ϵ it is given by Feynman's formula

$$\langle x|e^{-\epsilon H_{BS}}|x'\rangle = N(\epsilon)e^{\epsilon L_{BS}}$$

where $N(\epsilon)$ is a normalization constant. Since the formula (A4) is exact, we have, for $\delta x = x - x'$

$$L_{BS} = -\frac{1}{2\sigma^2}\left\{\frac{\delta x}{\epsilon} + r - \frac{1}{2}\sigma^2\right\}^2\tag{A5a}$$

and with

$$N(\epsilon) = \frac{1}{\sqrt{2\pi\epsilon\sigma^2}}\tag{A5b}$$

We now analyze the case of stochastic volatility. From eqn. (12b), the Hamiltonian is given by

$$H = -\frac{e^Y}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}e^Y - r\right) \frac{\partial}{\partial x} - \xi \rho e^{Y/2} \frac{\partial^2}{\partial x \partial y} - \frac{\xi^2}{2} \frac{\partial^2}{\partial y^2} + \left(\frac{1}{2}\xi^2 - \mu\right) \frac{\partial}{\partial y} \quad (A6a)$$

Hence we obtain using (A3)

$$\begin{aligned} \langle x, y | e^{-\epsilon H} | x', y' \rangle &= \int_{-\infty}^{\infty} \frac{dp_X}{2\pi} \frac{dp_Y}{2\pi} \langle x, y | e^{-\epsilon H} | p_X, p_Y \rangle \langle p_X, p_Y | x', y' \rangle \\ &= \int_{-\infty}^{\infty} \frac{dp_X}{2\pi} \frac{dp_Y}{2\pi} e^{ip_X(x-x')} e^{ip_Y(y-y')} e^{-\epsilon H(x, y, p_X, p_Y)} \end{aligned} \quad (A6b)$$

and from (A4) the matrix elements of the Hamiltonian is given by

$$H(x, y, p_X, p_Y) = \frac{e^Y}{2} p_x^2 + \left(\frac{1}{2}e^Y - r\right) ip_X + \xi \rho e^{Y/2} p_X p_Y + \frac{\xi^2}{2} p_Y^2 + \left(\frac{1}{2}\xi^2 - \mu\right) ip_Y \quad (A7)$$

To perform the Gaussian integration over p_X and p_Y we need the inverse and the determinant of the matrix

$$M = \begin{bmatrix} e^Y & \xi \rho e^{Y/2} \\ \xi \rho e^{Y/2} & \xi^2 \end{bmatrix} \quad (A8)$$

Note

$$\det M = \xi^2 e^Y (1 - \rho^2) \quad (A9)$$

and

$$M^{-1} = \frac{1}{\xi^2(1 - \rho^2)} \begin{bmatrix} \xi^2 e^{-Y} & -\xi \rho e^{-Y/2} \\ -\xi \rho e^{-Y/2} & 1 \end{bmatrix} \quad (A9)$$

Let

$$A = x - x' + \epsilon r - \frac{\epsilon}{2} e^Y \quad (A11)$$

and

$$B = y - y' + \epsilon \mu - \frac{\epsilon}{2} \xi^2 \quad (A12)$$

From (A6) and (A7), on performing the Gaussian integrations, we have the Feynman relation

$$\langle x, y | e^{-\epsilon H} | x', y' \rangle = \frac{1}{2\pi \epsilon \sqrt{\det M}} e^{\epsilon L} \quad (A13)$$

where from (A6), (A8) and (A10)

$$L = -\frac{1}{2\epsilon^2(1 - \rho^2)} (e^{-Y} A^2 + \frac{1}{\xi^2} B^2 - 2\frac{\rho}{\xi} e^{-Y/2} AB) + O(\epsilon) \quad (A14)$$

and finally from (A11) and (A12), for $\delta x = x - x'$ and $\delta y = y - y'$, we have the negative definite lagrangian given by

$$L = - \frac{1}{2\xi^2} \left(\frac{\delta y}{\epsilon} + \mu - \frac{1}{2}\xi^2 \right)^2 - \frac{e^{-Y}}{2(1-\rho^2)} \left[\frac{\delta x}{\epsilon} + r - \frac{1}{2}e^Y - \frac{\rho}{\xi}e^{Y/2} \left(\frac{\delta y}{\epsilon} + \mu - \frac{1}{2}\xi^2 \right) \right]^2 + O(\epsilon) \quad (A15)$$

Appendix 3: The $\int DX$ - Path Integration

The path integration over the $x(t)$ -variables in eqn. (19) can be done exactly, and the derivation is given below [9].

Let

$$Q = \int DX e^{S_X} \quad (A16)$$

where from (22b) the x -dependent term of the Lagrangian is given by

$$L_X(i) = - \frac{e^{-Y_i}}{2(1-\rho^2)} \left\{ \frac{\delta x_i}{\epsilon} + r - \frac{1}{2}e^{Y_i} - \frac{\rho}{\xi}e^{Y_i/2} \left(\frac{\delta y_i}{\epsilon} + \mu - \frac{1}{2}\xi^2 \right) \right\}^2 + O(\epsilon) \quad (A16b)$$

Let

$$c_i = r - \frac{1}{2}e^{Y_i} - \frac{\rho}{\xi}e^{Y_i/2} \left(\frac{\delta y_i}{\epsilon} + \mu - \frac{1}{2}\xi^2 \right) \quad (A17a)$$

Then

$$S_X = - \frac{1}{2\epsilon(1-\rho^2)} \sum_{i=1}^N e^{-Y_i} (x_i - x_{i-1} + \epsilon c_i)^2 \quad (A17b)$$

with boundary values given by

$$x_0 = x', x_N = x \quad (A17c)$$

We make the change of variables

$$x_i = z_i - \epsilon \sum_{j=1}^i c_j, \quad dx_i = dz_i, \quad i = 1, 2, \dots, N-1 \quad (A18a)$$

with boundary values

$$z_0 = x' \quad (A18b)$$

$$z_N = x + \epsilon \sum_{j=1}^N c_j \quad (A18c)$$

We hence have from (A17b) and (A18a)

$$S_Z = - \frac{1}{2\epsilon(1-\rho^2)} \sum_{i=1}^N e^{-Y_i} (z_i - z_{i-1})^2 \quad (A19a)$$

and, from (25c) and (A18a)

$$Q = \frac{e^{-Y_N/2}}{\sqrt{2\pi\epsilon(1-\rho^2)}} \prod_{i=1}^{N-1} \int_{-\infty}^{+\infty} \frac{dz_i e^{-Y_i/2}}{\sqrt{2\pi\epsilon(1-\rho^2)}} e^{S_Z} \quad (A19b)$$

All the z_i integrations can be performed exactly; one starts from the boundary by first integrating over z_1 , and then over z_2 ,and finally over z_{N-1} . Consider the z_1 -integration. We have

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{dz_1 e^{-Y_1/2}}{\sqrt{2\pi\epsilon(1-\rho^2)}} \exp\left\{-\frac{1}{2\epsilon(1-\rho^2)}[e^{-Y_2}(z_2 - z_1)^2 + e^{-Y_1}(z_1 - z_0)^2]\right\} \\ &= \frac{e^{Y_2/2}}{\sqrt{e^{Y_1} + e^{Y_2}}} \exp\left\{-\frac{1}{2\epsilon(1-\rho^2)} \frac{1}{e^{Y_1} + e^{Y_2}} (z_2 - z_0)^2\right\} \end{aligned} \quad (A20)$$

Repeating this procedure $(N-1)$ times yields

$$Q = \frac{e^{S_1}}{\sqrt{2\pi\epsilon(1-\rho^2) \sum_{i=1}^N e^{Y_i}}} \quad (A21a)$$

where

$$\begin{aligned} S_1 &= -\frac{1}{2\epsilon(1-\rho^2) \sum_{i=1}^N e^{Y_i}} (z_N - z_0)^2 \\ &= -\frac{1}{2\epsilon(1-\rho^2) \sum_{i=1}^N e^{Y_i}} \left\{x - x' + \epsilon \sum_{i=1}^N c_i\right\}^2 \end{aligned} \quad (A21b)$$

For the case of constant volatility $\xi = 0 = \rho$ and $e^{Y_i} = \sigma^2 = \text{constant}$ for all i ; eqn. (A21b) then reduces to the Black- Scholes given in eqn. (A3).

Appendix 4 : Generating Function $Z(j, y, p)$

We evaluate

$$Z(j, y, p) = \int DY \exp\left[\int_0^\tau dt j(t)y(t)\right] e^{pY(0)/2} e^{S_0}; \quad (A22a)$$

from (27b) we have

$$S_0 = -\frac{1}{2\xi^2} \int_0^\tau dt \left(\frac{dy}{dx} + \mu - \frac{1}{2}\xi^2\right)^2 \quad (Ab)$$

with boundary condition

$$y(\tau) = y. \quad (A22c)$$

We first evaluate $Z(j, y, y')$ given by the path integral in (A22a) but with the boundary condition

$$y(0) = y', y(\tau) = y \quad (A23)$$

and from eqn. (A22a)

$$Z(j, y, p) = \int_{-\infty}^{\infty} dy' Z(j, y, y') e^{py'/2}. \quad (A24)$$

Define new path integration variables $z(t)$ by

$$z(t) = y(t) - y' + \frac{t}{\tau}(y' - y). \quad (A25)$$

Note the new variables $z(t)$ due to (A23) have boundary conditions

$$z(0) = 0 = z(\tau). \quad (A26)$$

Hence

$$Z(j, y, y') = e^{W_0} \int DZ e^{S_Z} \quad (A27)$$

with

$$\begin{aligned} W_0 = & -\frac{1}{2\tau\xi^2}(y' - y - \mu\tau + \frac{1}{2}\xi^2\tau^2)^2 + y' \int_0^\tau dt j(t) \\ & - \frac{y' - y}{\tau} \int_0^\tau dt t j(t) \end{aligned} \quad (A28)$$

and

$$S_Z = \int_0^\tau dt j(t) z(t) - \frac{1}{2\xi^2} \int_0^\tau dt \left(\frac{dz}{dt}\right)^2. \quad (A29)$$

To perform the path integral over $z(t)$, note that from boundary condition (A26) we have the Fourier sine expansion

$$z(t) = \sum_{n=1}^{\infty} \sin(\pi n t / \tau) z_n \quad (A30)$$

From (A29) and (A30) we have

$$S_Z = -\frac{\pi^2}{4\tau\xi^2} \sum_{n=1}^{\infty} n^2 z_n^2 + \sum_{n=1}^{\infty} \left[\int_0^\tau dt j(t) \sin(\pi n t / \tau) \right] z_n \quad (A31)$$

The path integration over $z(t)$ factorizes into infinitely many Gaussian integrations over the z_n 's and we obtain

$$\int DZ e^{S_Z} = C' \int_{-\infty}^{\infty} dz_1 dz_2 dz_3 \cdots dz_{\infty} e^{S_Z} \quad (A32)$$

$$= C(\tau) e^W \quad (A33)$$

where

$$W = \frac{\xi^2 \tau}{\pi^2} \int_0^\tau dt dt' j(t) D(t, t') j(t') \quad (A34)$$

$$= \frac{\xi^2}{\tau} \int_0^\tau dt \int_0^t dt' j(t) (\tau - t) t' j(t') \quad (A35)$$

since

$$D(t, t') = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\pi n t / \tau) \sin(\pi n t' / \tau) \quad (A36)$$

$$= \frac{\pi^2}{2\tau} [t' \theta(t - t') + t \theta(t' - t) - \frac{t t'}{\tau}]. \quad (A37)$$

where the step function is defined by

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases} \quad (A37a)$$

The normalization function $C(\tau)$ can be evaluated by first considering the discrete and finite version of the DZ - path integral along the lines discussed in Section III. The result is given [Ref 9] by

$$C(\tau) = \frac{1}{\sqrt{2\pi\xi^2\tau}} \quad (A38)$$

Collecting our results we have

$$Z(j, y', y) = \frac{e^{W_0 + W}}{\sqrt{2\pi\xi^2\tau}} \quad (A39)$$

and performing the y' Gaussian integration in (A24) finally yields

$$Z(j, y, p) = e^F \quad (A40)$$

with

$$\begin{aligned} F = & y \int_0^\tau dt j(t) + \left(\mu - \frac{1}{2}\xi^2\right) \int_0^\tau dt (\tau - t) j(t) \\ & + \xi^2 \int_0^\tau dt j(t) (\tau - t) \int_0^t dt' j(t') + F' \end{aligned} \quad (A41)$$

where

$$F' = \frac{1}{2} y p + \frac{1}{2} \xi^2 p \int_0^\tau dt (\tau - t) j(t) + \frac{1}{2} p \tau \left(\mu - \frac{1}{2}\xi^2\right) + \frac{1}{8} p^2 \xi^2 \tau. \quad (A42)$$